ESC103 Unit 17

Aspen Erlandsson

November 22, 2022

Abstract

1 Numerical Solutions to First Order Differential Equations

Initial Value Problems (IVPs)

Example:

Given:

$$\frac{dy}{dx} = y'(t) = cy(t)$$

c is a scalar, and:

$$y(t = 0) = y(0)$$

Can we find $y(t)$ that will satisfy the D.E. (Differential Equation), and the I.C. (Initial Condition)

Let's propose a solution and check it:

$$y(t) = y(0)e^{ct}$$

$$y'(t) = cy(0) = cy(t)$$

Satisfied!

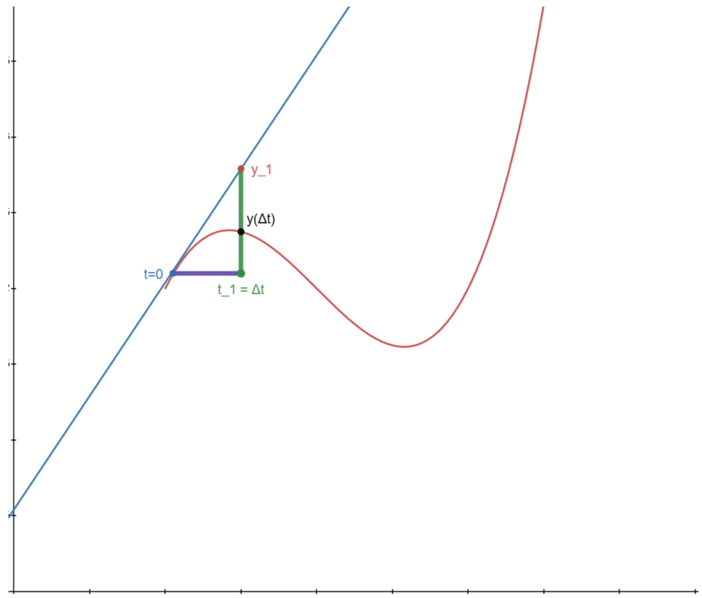
$$y(t = 0) = y(0)e^{c(0)} = y(0)$$

Euler's Method: Generally, given:

$$y'(t) = f(t, y, (t))$$

and

$$y(t = 0) = y(0)$$



Some unknown solution curve, with the blue line being our intial known "Slope" at the beginning $= y'(t = 0, y(0))$

$$y_1 = y(0) + \boldsymbol{\Delta} t y'(t = 0)$$

$$y_1 = y(0) + \boldsymbol{\Delta} t f(0, y(0))$$

Algorithm:

$$t_{n+1} = t_n + \boldsymbol{\Delta} t$$

$$y_{n+1} = y_n + \boldsymbol{\Delta} t f(t_n, y_n)$$

Where $y_1, y_2, y_3...$ are numerical estimates of the solution.

$$y(t_1 = \boldsymbol{\Delta} t), y(t_2 = 2\boldsymbol{\Delta} t), y(t_3 = \boldsymbol{\Delta} t)$$

Sources of error with this method:

S1: The formula used to go from (t_n, y_n) to (t_{n+1}, y_{n+1}) is not exact.

S2: The information going into the formula is not exact because (t_n, y_n) does not lie on the true solution curve except the first point.

Improved Eulers Method

$$t_{n+1} = t_n + \boldsymbol{\Delta} t$$

$$y_{n+1} = y_n + \boldsymbol{\Delta} t S$$

where:

$$S = \frac{1}{2}(S_L + S_R)$$

$$S_L = \text{ slope est at } t_n$$

$$S_R = \text{ slope est at } t_{n+1}$$

A solution is to replace y_{n+1} at the beginning by its estimate obtained using Eulers method.

Algorithm:

$$t_{n+1} = t_n + \boldsymbol{\Delta} t$$

$$y_{n+1,Em} = y_{n,IEm} + \boldsymbol{\Delta} t f(t_n, y_{n,IEm})$$

$$y_{n+1,IEm} = y_{n,IEm} + \frac{\boldsymbol{\Delta} t}{2}(f(t_n y_{n,IEm}) + f(t_{n+1}, y_{n+1,Em}))$$

My understanding: Essentially: step forward using the KNOWN slope, figure out what the predicted y is, using that calculate what the slope should be (since we have derivative in terms of the function), but don't actually move there, then we take the average between the slope at the start, and slope that we just found and the stepped point. THEN we ACTUALLY move our position to the result of stepping that averaged slope.